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# Ultradiscrete Analogue of the Identity of Pfaffians

By

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## Contents



## Abstract

We present an algebraic identity of ultradiscretized hafnians, which is an ultradiscrete form of the identity of pfaffians. The identity stems from a decomposition of a product of the hafnians.

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## § 1. Introduction

Recent progress in the direct method in soliton theory reveals that a soliton equation which exhibits multi-soliton solution is reduced to an identity of pfaffians. The identity of determinants such as the Plücker relation and Jacobi's identity of determinants are special cases of the identity of the pfaffians [1]. Takahashi and the author of the present article have shown that soliton solutions to the box and ball system follow a form of ultradiscretized *permanent* [2]. A permanent is a signature free determinant. Nagai has shown that soliton solutions to the ultradiscrete Toda equation are expressed also by the ultradiscretized permanents [3]. These facts suggest that there must be an identity of ultradiscretized permanents instead of determinants. More generally we expect an identity of ultradiscretized hafnians instead of pfaffians. A hafnian is a signature free pfaffian introduced by Caieniello [4].

We look for the ultradiscretization of the following simple identity of determinants

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix} + \begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \end{vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = 0, \tag{1.1}$$

which is one of the Plücker relations.

Let each term in Eq.(1.1) be $p_1, p_2$ and $p_3$,namely

$$p_1 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix} = a_1a_3b_2b_4 - a_1a_4b_2b_3 - a_2a_3b_1b_4 + a_2a_4b_1b_3, \tag{1.2}$$

$$p_2 = \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix} = a_1a_2b_3b_4 - a_1a_4b_2b_3 - a_2a_3b_1b_4 + a_3a_4b_1b_2, \tag{1.3}$$

$$p_3 = \begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \end{vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = a_1a_2b_3b_4 - a_1a_3b_2b_4 - a_2a_4b_1b_3 + a_3a_4b_1b_2. \tag{1.4}$$

Then the Plücker relation is written as

$$p_1 - p_2 + p_3 = 0, \tag{1.5}$$

which cannot be ultradiscretized because of negative terms in $p_1, p_2$ and $p_3$.

Now we replace the determinants by the corresponding permanents

$$q_1 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}_+ \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix}_+ = a_1a_3b_2b_4 + a_1a_4b_2b_3 + a_2a_3b_1b_4 + a_2a_4b_1b_3, \tag{1.6}$$

$$q_2 = \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}_+ \begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix}_+ = a_1a_2b_3b_4 + a_1a_4b_2b_3 + a_2a_3b_1b_4 + a_3a_4b_1b_2, \tag{1.7}$$

$$q_3 = \begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \end{vmatrix}_+ \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}_+ = a_1a_2b_3b_4 + a_1a_3b_2b_4 + a_2a_4b_1b_3 + a_3a_4b_1b_2, \tag{1.8}$$

where $q_1, q_2$ and $q_3$ have no negative terms and can be ultradiscretized. However the corresponding Plücker relation does not hold,

$$q_1 - q_2 + q_3 = 2(a_1a_3b_2b_4 + a_2a_4b_1b_3) \neq 0, \tag{1.9}$$

which indicates that ultradiscrete analogue of the Plücker relation is not easy to find.

We notice comparing the expressions for the products of the determinants, $p_1, p_2$ and $p_3$ that they are decomposed into a sum of common terms $p_{12}, p_{13}$ and $p_{23}$ where $p_{ij}$ is the common term of $p_i$ and $p_j$ for $i, j = 1, 2, 3$,

$$p_1 = -p_{12} + p_{13}, \quad p_2 = -p_{12} + p_{23}, \quad p_3 = -p_{13} + p_{23}, \tag{1.10}$$

where

$$p_{12} = a_1a_4b_2b_3 + a_2a_3b_1b_4, \tag{1.11}$$

$$p_{13} = a_1a_3b_2b_4 + a_2a_4b_1b_3, \tag{1.12}$$

$$p_{23} = a_1a_2b_3b_4 + a_3a_4b_1b_2. \tag{1.13}$$

The permanents have the same terms as the determinants except the signature. Accordingly we find that the products of the permanents, $q_1, q_2$ and $q_3$ are decomposed into a sum of common terms $q_{12}, q_{13}$ and $q_{23}$,where $q_{ij}$ is the common term of $q_i$ and $q_j$ for $i, j = 1, 2, 3$,

$$q_1 = q_{12} + q_{13}, \quad q_2 = q_{12} + q_{23}, \quad q_3 = q_{13} + q_{23}, \tag{1.14}$$

where

$$q_{12} = a_1a_4b_2b_3 + a_2a_3b_1b_4, \tag{1.15}$$

$$q_{13} = a_1a_3b_2b_4 + a_2a_4b_1b_3, \tag{1.16}$$

$$q_{23} = a_1a_2b_3b_4 + a_3a_4b_1b_2. \tag{1.17}$$

The Plücker relation Eq.(1.1) is comfirmed by the decomposition of products of determinanats,

$$p_1 - p_2 + p_3 = -p_{12} + p_{13} - (-p_{12} + p_{23}) - p_{13} + p_{23} = 0. \tag{1.18}$$

An ultradiscrete analogue of the Plücker relation is obtained as follows.

Replacing the determinants by the correspoding permanents we have

$$q_1 + q_3 = q_2. \tag{1.19}$$

Let

$$q_i = \exp(Q_i/\epsilon) \text{ for } i = 1, 2, 3, \tag{1.20}$$

$$q_{ij} = \exp(Q_{ij}/\epsilon) \text{ for } i, j = 1, 2, 3, \tag{1.21}$$

where $\epsilon$ is the ultradiscrete parameter [5]. In the small limit of $\epsilon$ we have an ultradiscrete analogue of the Plücker relation , Eq.(1.19),

$$Q_2 = \max(Q_1, Q_3), \tag{1.22}$$

which does not hold in general.

We investigate under what conditions on $Q_1, Q_2$ and $Q_3$ Eq.(1.22) does hold. The ultradiscrete form of Eq.(1.14) are

$$\begin{aligned} Q_1 &= \max(Q_{12}, Q_{13}),\\ Q_2 &= \max(Q_{12}, Q_{23}),\\ Q_3 &= \max(Q_{13}, Q_{23}). \end{aligned} \tag{1.23}$$

Substituting these expressions into Eq.(1.22) we obtain

$$\max(Q_{12}, Q_{23}) = \max(Q_{12}, Q_{13}, Q_{23}). \tag{1.24}$$

Obviously Eq.(1.24) does hold if $Q_{13} \leq \max(Q_{12}, Q_{23})$. But it does not hold if $Q_{13} > \max(Q_{12}, Q_{23})$.

However if $Q_{13} > \max(Q_{12}, Q_{23})$ we find, using Eq.(1.23)

$$Q_1 = Q_3. \tag{1.25}$$

Hence we obtain the following algebraic identity of the ultradiscretized permanents,

$$[Q_2 - \max(Q_1, Q_3)](Q_1 - Q_3) = 0, \tag{1.26}$$

which we call "ultradiscrete analogue of the Plücker relation".

The contents of this article are as follows. In §2,we describe the fundamental properties of determinants,pfaffians,permanents and hafnians. In §3 we prove by induction a decomposition of a product of hafnians into a sum of common terms, which will be used in §4 to obtain an algebraic identity of ultradiscretized hafnians that is a ultradiscrete analogue of the identity of pfaffians.

## § 2. Preliminaries

In this section we describe elementary properties of determinants, permanents, pfaffians and hafnians.

### § 2.1. Determinants, pfaffians, permanents and hafnians

Consider an $n \times n$ matrix $\boldsymbol{D} = (x_{ij})$ whose determinant is $D$. We write the $n$th-order determinant $D$ as

$$D = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix}_{-} . \tag{2.1}$$

An $n$th-order pfaffian denoted by $\mathrm{pf}(1, 2, \cdots, 2n)$ is anti-symmetric with respect to the indices, $1, 2, \cdots, 2n$, namely

$$\mathrm{pf}(1, 2, \cdots, i, \cdots, j, \cdots, 2n) = -\mathrm{pf}(1, 2, \cdots, j, \cdots, i, \cdots, 2n) \tag{2.2}$$

and is defined by the expansion formula

$$\mathrm{pf}(1, 2, \cdots, 2n) = \sum_{j=2}^{2n} \mathrm{pf}(1, j)(-1)^j \mathrm{pf}(2, 3, \cdots, \hat{j}, \cdots, 2n), \tag{2.3}$$

where $\hat{j}$ means that index $j$ is omitted.

We have, for example,

$$\mathrm{pf}(1, 2, 3, 4) = \mathrm{pf}(1, 2)\mathrm{pf}(3, 4) - \mathrm{pf}(1, 3)\mathrm{pf}(2, 4) + \mathrm{pf}(1, 4)\mathrm{pf}(2, 3). \tag{2.4}$$

An $n$th-order determinant $D$ is expressed by an $n$th-order pfaffian as follows

$$D = \mathrm{pf}(x_1, x_2, \cdots, x_n, n, n-1, \cdots, 2, 1) \tag{2.5}$$

$$= (-1)^{n(n-1)/2} \mathrm{pf}(x_1, x_2, \cdots, x_n, 1, 2, \cdots, n), \tag{2.6}$$

where the pfaffian entries are defined by

$$\mathrm{pf}(x_i, x_j) = 0, \quad \mathrm{pf}(x_i, j) = x_{ij}, \quad \text{for } i, j = 1, 2, \cdots, n. \tag{2.7}$$

A permanent is a signature free determinant, that is the signatures of permutations are not taken into account in its definition.

The permanent of an $n \times n$ matrix $\boldsymbol{D} = (x_{ij})$ is defined as

$$perm(D) = \sum_{\sigma \in S} \prod_{i=1}^{n} x_{i,\sigma(i)}, \tag{2.8}$$

where the sum extends over all permutations of number $1, 2, \cdots, n$. We write the $n$th-order permanent $P$ as

$$P = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix}_{+} . \tag{2.9}$$

An $n$th-order hafnian denoted by $\mathrm{hf}(1,2,\cdots,2n)$ is a signature free pfaffian, which is symmetric with respect to the indices, $1,2,\cdots,2n$ and is defined by the expansion formula

$$\mathrm{hf}(1,2,\cdots,2n)=\sum_{j=2}^{2n}\mathrm{hf}(1,j)\mathrm{hf}(2,3,\cdots,\hat{j},\cdots,2n). \tag{2.10}$$

We have, for example,

$$\mathrm{hf}(1,2,3,4)=\mathrm{hf}(1,2)\mathrm{hf}(3,4)+\mathrm{hf}(1,3)\mathrm{hf}(2,4)+\mathrm{hf}(1,4)\mathrm{hf}(2,3). \tag{2.11}$$

The $n$th-order permanent $P$ is expressed by the $n$th-order hafnian.

$$P=\mathrm{hf}(x_1,x_2,\cdots,x_n,1,2,\cdots,n), \tag{2.12}$$

if one define the hafnian entries by

$$\mathrm{hf}(x_i,x_j)=0,\quad \mathrm{hf}(x_i,j)=x_{ij},\quad \text{for } i,j=1,2,\cdots,n. \tag{2.13}$$

## § 2.2. Laplace expansion of permanents

Consider the permanent $P_4$ of an $4\times 4$ matrix $\boldsymbol{D}=(a_{ij})$,

$$P_4=\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{vmatrix}_+ . \tag{2.14}$$

We have the Laplace expansion formula of $P_4$,

$$\begin{aligned} P_4=&\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}_+\begin{vmatrix} a_{33} & a_{34}\\ a_{43} & a_{44}\end{vmatrix}_+ + \begin{vmatrix} a_{13} & a_{14}\\ a_{23} & a_{24}\end{vmatrix}_+\begin{vmatrix} a_{31} & a_{32}\\ a_{41} & a_{42}\end{vmatrix}_+\\ &+\begin{vmatrix} a_{11} & a_{13}\\ a_{21} & a_{23}\end{vmatrix}_+\begin{vmatrix} a_{32} & a_{34}\\ a_{42} & a_{44}\end{vmatrix}_+ + \begin{vmatrix} a_{12} & a_{14}\\ a_{22} & a_{24}\end{vmatrix}_+\begin{vmatrix} a_{31} & a_{33}\\ a_{41} & a_{43}\end{vmatrix}_+\\ &+\begin{vmatrix} a_{11} & a_{14}\\ a_{21} & a_{24}\end{vmatrix}_+\begin{vmatrix} a_{32} & a_{33}\\ a_{42} & a_{43}\end{vmatrix}_+ + \begin{vmatrix} a_{12} & a_{13}\\ a_{22} & a_{23}\end{vmatrix}_+\begin{vmatrix} a_{31} & a_{34}\\ a_{41} & a_{44}\end{vmatrix}_+ , \end{aligned} \tag{2.15}$$

which is expressed by the 4th-order hafnians with the entries, $\mathrm{hf}(y_i,y_j)=0$ and $\mathrm{hf}(y_i,j)=a_{i,j}$ for $i,j=1,2,3,4$,

$$\begin{aligned} P_4&=\mathrm{hf}(y_1,y_2,y_3,y_4,1,2,3,4)\\ &=\mathrm{hf}(y_1,y_2,1,2)\mathrm{hf}(y_3,y_4,3,4)+\mathrm{hf}(y_1,y_2,3,4)\mathrm{hf}(y_3,y_4,1,2)\\ &+\mathrm{hf}(y_1,y_2,1,3)\mathrm{hf}(y_3,y_4,2,4)+\mathrm{hf}(y_1,y_2,2,4)\mathrm{hf}(y_3,y_4,1,3)\\ &+\mathrm{hf}(y_1,y_2,1,4)\mathrm{hf}(y_3,y_4,2,3)+\mathrm{hf}(y_1,y_2,2,3)\mathrm{hf}(y_3,y_4,1,4). \end{aligned} \tag{2.16}$$

Interchanging the index $y_2$ with the index $y_3$ in (2.16) we obtain

$$\begin{aligned} P_4 &= \mathrm{hf}(y_1, y_3, y_2, y_4, 1, 2, 3, 4) \\ &= \mathrm{hf}(y_1, y_3, 1, 2)\mathrm{hf}(y_2, y_4, 3, 4) + \mathrm{hf}(y_1, y_3, 3, 4)\mathrm{hf}(y_2, y_4, 1, 2) \\ &\quad +\mathrm{hf}(y_1, y_3, 1, 3)\mathrm{hf}(y_2, y_4, 2, 4) + \mathrm{hf}(y_1, y_3, 2, 4)\mathrm{hf}(y_2, y_4, 1, 3) \\ &\quad +\mathrm{hf}(y_1, y_3, 1, 4)\mathrm{hf}(y_2, y_4, 2, 3) + \mathrm{hf}(y_1, y_3, 2, 3)\mathrm{hf}(y_2, y_4, 1, 4). \end{aligned} \tag{2.17}$$

Similarly interchanging the index $y_2$ with the index $y_4$ in (2.16) we obtain

$$\begin{aligned} P_4 &= \mathrm{hf}(y_1, y_4, 1, 2)\mathrm{hf}(y_2, y_3, 3, 4) + \mathrm{hf}(y_1, y_4, 3, 4)\mathrm{hf}(y_2, y_3, 1, 2) \\ &\quad +\mathrm{hf}(y_1, y_4, 1, 3)\mathrm{hf}(y_2, y_3, 2, 4) + \mathrm{hf}(y_1, y_4, 2, 4)\mathrm{hf}(y_2, y_3, 1, 3) \\ &\quad +\mathrm{hf}(y_1, y_4, 1, 4)\mathrm{hf}(y_2, y_3, 2, 3) + \mathrm{hf}(y_1, y_4, 2, 3)\mathrm{hf}(y_2, y_3, 1, 4). \end{aligned} \tag{2.18}$$

Permanent $P_4$ is invariant under changing the rows and columns,

$$P_4 = \begin{vmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{vmatrix}_+ . \tag{2.19}$$

Then the Laplace expansion formula of $P_4$ becomes

$$\begin{aligned} P_4 = &\begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix}_+ \begin{vmatrix} a_{33} & a_{43} \\ a_{34} & a_{44} \end{vmatrix}_+ + \begin{vmatrix} a_{31} & a_{41} \\ a_{32} & a_{42} \end{vmatrix}_+ \begin{vmatrix} a_{13} & a_{23} \\ a_{14} & a_{24} \end{vmatrix}_+ \\ + &\begin{vmatrix} a_{11} & a_{31} \\ a_{12} & a_{32} \end{vmatrix}_+ \begin{vmatrix} a_{23} & a_{43} \\ a_{24} & a_{44} \end{vmatrix}_+ + \begin{vmatrix} a_{21} & a_{41} \\ a_{22} & a_{42} \end{vmatrix}_+ \begin{vmatrix} a_{13} & a_{33} \\ a_{14} & a_{34} \end{vmatrix}_+ \\ + &\begin{vmatrix} a_{11} & a_{41} \\ a_{12} & a_{42} \end{vmatrix}_+ \begin{vmatrix} a_{23} & a_{33} \\ a_{24} & a_{34} \end{vmatrix}_+ + \begin{vmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{vmatrix}_+ \begin{vmatrix} a_{13} & a_{43} \\ a_{14} & a_{44} \end{vmatrix}_+ , \end{aligned} \tag{2.20}$$

which is expressed by the 4th-order hafnians with the entries, $\mathrm{hf}(y_i, y_j) = 0$ and $\mathrm{hf}(y_i, j) = a_{ij}$ for $i, j = 1, 2, 3, 4$,

$$\begin{aligned} P_4 &= \mathrm{hf}(y_1, y_2, y_3, y_4, 1, 2, 3, 4) \\ &= \mathrm{hf}(y_1, y_2, 1, 2)\mathrm{hf}(y_3, y_4, 3, 4) + \mathrm{hf}(y_1, y_2, 3, 4)\mathrm{hf}(y_3, y_4, 1, 2) \\ &\quad +\mathrm{hf}(y_1, y_3, 1, 2)\mathrm{hf}(y_2, y_4, 3, 4) + \mathrm{hf}(y_1, y_3, 3, 4)\mathrm{hf}(y_2, y_4, 1, 2) \\ &\quad +\mathrm{hf}(y_1, y_4, 1, 2)\mathrm{hf}(y_2, y_3, 3, 4) + \mathrm{hf}(y_1, y_4, 3, 4)\mathrm{hf}(y_2, y_3, 1, 2). \end{aligned} \tag{2.21}$$

Interchanging the index 2 with the index 4 in (2.21) we obtain

$$\begin{aligned} P_4 &= \mathrm{hf}(y_1, y_2, 1, 4)\mathrm{hf}(y_3, y_4, 2, 3) + \mathrm{hf}(y_1, y_2, 2, 3)\mathrm{hf}(y_3, y_4, 1, 4) \\ &\quad +\mathrm{hf}(y_1, y_3, 1, 4)\mathrm{hf}(y_2, y_4, 2, 3) + \mathrm{hf}(y_1, y_3, 2, 3)\mathrm{hf}(y_2, y_4, 1, 4) \\ &\quad +\mathrm{hf}(y_1, y_4, 1, 4)\mathrm{hf}(y_2, y_3, 2, 3) + \mathrm{hf}(y_1, y_4, 2, 3)\mathrm{hf}(y_2, y_3, 1, 4). \end{aligned} \tag{2.22}$$

Subtracting the terms, $\mathrm{hf}(y_1, y_2, 1, 4)\mathrm{hf}(y_3, y_4, 2, 3) + \mathrm{hf}(y_1, y_2, 2, 3)\mathrm{hf}(y_3, y_4, 1, 4)$ from Eqs.(2.16) and (2.22) we obtain an identity of the hafnians,

$$\begin{aligned}
&\mathrm{hf}(y_1, y_2, 1, 2)\mathrm{hf}(y_3, y_4, 3, 4) + \mathrm{hf}(y_1, y_2, 3, 4)\mathrm{hf}(y_3, y_4, 1, 2)\\
&+\mathrm{hf}(y_1, y_2, 1, 3)\mathrm{hf}(y_3, y_4, 2, 4) + \mathrm{hf}(y_1, y_2, 2, 4)\mathrm{hf}(y_3, y_4, 1, 3)\\
&= \mathrm{hf}(y_1, y_3, 1, 4)\mathrm{hf}(y_2, y_4, 2, 3) + \mathrm{hf}(y_1, y_3, 2, 3)\mathrm{hf}(y_2, y_4, 1, 4)\\
&+\mathrm{hf}(y_1, y_4, 1, 4)\mathrm{hf}(y_2, y_3, 2, 3) + \mathrm{hf}(y_1, y_4, 2, 3)\mathrm{hf}(y_2, y_3, 1, 4).
\end{aligned}\tag{2.23}$$

Interchanging the index 3 with the index 4 in (2.23) we obtain an identity,

$$\begin{aligned}
&\mathrm{hf}(y_1, y_2, 1, 2)\mathrm{hf}(y_3, y_4, 3, 4) + \mathrm{hf}(y_1, y_2, 3, 4)\mathrm{hf}(y_3, y_4, 1, 2)\\
&+\mathrm{hf}(y_1, y_2, 1, 4)\mathrm{hf}(y_3, y_4, 2, 3) + \mathrm{hf}(y_1, y_2, 2, 3)\mathrm{hf}(y_3, y_4, 1, 4)\\
&= \mathrm{hf}(y_1, y_3, 1, 3)\mathrm{hf}(y_2, y_4, 2, 4) + \mathrm{hf}(y_1, y_3, 2, 4)\mathrm{hf}(y_2, y_4, 1, 3)\\
&+\mathrm{hf}(y_1, y_4, 1, 3)\mathrm{hf}(y_2, y_3, 2, 4) + \mathrm{hf}(y_1, y_4, 2, 4)\mathrm{hf}(y_2, y_3, 1, 3).
\end{aligned}\tag{2.24}$$

Similarly interchanging the index 2 with the index 3 in (2.24) we obtain an identity,

$$\begin{aligned}
&\mathrm{hf}(y_1, y_2, 1, 3)\mathrm{hf}(y_3, y_4, 2, 4) + \mathrm{hf}(y_1, y_2, 2, 4)\mathrm{hf}(y_3, y_4, 1, 3)\\
&+\mathrm{hf}(y_1, y_2, 1, 4)\mathrm{hf}(y_3, y_4, 2, 3) + \mathrm{hf}(y_1, y_2, 2, 3)\mathrm{hf}(y_3, y_4, 1, 4)\\
&= \mathrm{hf}(y_1, y_3, 1, 2)\mathrm{hf}(y_2, y_4, 3, 4) + \mathrm{hf}(y_1, y_3, 3, 4)\mathrm{hf}(y_2, y_4, 1, 2)\\
&+\mathrm{hf}(y_1, y_4, 1, 2)\mathrm{hf}(y_2, y_3, 3, 4) + \mathrm{hf}(y_1, y_4, 3, 4)\mathrm{hf}(y_2, y_3, 1, 2).
\end{aligned}\tag{2.25}$$

These identities will be used in the section 3 in order to decompose $\hat{h}_1$,$\hat{h}_2$ and $\hat{h}_3$.

## § 2.3. Plücker relations

Let $a_i$ for $i = 1, 2, \cdots, N-1$ and $b_i$ for $i = 1, 2, \cdots, n$ be $N$-dimensional vectors and $|\cdots|$ express determinants. The Plücker relations are expressed in general by

$$\begin{aligned}
&\sum_{i=1}^{n}(-1)^i|a_1, a_2, \cdots, a_{N-1}, b_i|\\
&\qquad\times|a_1, a_2, \cdots, a_{N-n+1}, b_n, b_{n-1}, \cdots, \hat{b}_i, \cdots, b_1| = 0,
\end{aligned}\tag{2.26}$$

where $\hat{b}$ means that symbol $b$ is omitted.

We consider the simplest case, namely $n = 3$ and write the Plücker relation as,

$$D_{ab}D_{cd} - D_{ac}D_{bd} + D_{ad}D_{bc} = 0,\tag{2.27}$$

where $D_{ab}, D_{cd}, D_{ac}, D_{bd}, D_{ad}$ and $D_{bc}$ are $n$th-order determinants defined by

$$D_{ab} = |a, b, x_3, x_4, \cdots, x_n|, \tag{2.28}$$

$$D_{ac} = |a, c, x_3, x_4, \cdots, x_n|, \tag{2.29}$$

$$D_{ad} = |a, d, x_3, x_4, \cdots, x_n|, \tag{2.30}$$

$$D_{bc} = |b, c, x_3, x_4, \cdots, x_n|, \tag{2.31}$$

$$D_{bd} = |b, d, x_3, x_4, \cdots, x_n|, \tag{2.32}$$

$$D_{cd} = |c, d, x_3, x_4, \cdots, x_n|, \tag{2.33}$$

where $a, b, c, d$ and $x_i (i = 3, 4, \cdots, n)$ are $n$-dimensional vectors.

## § 2.4. Identity of pfaffians

We have the following identity of pfaffians,

$$\begin{aligned} \mathrm{pf}(1,2,\cdots,2n)\mathrm{pf}(5,6,\cdots,2n) = {} & \mathrm{pf}(1,2,5,6,\cdots,2n)\mathrm{pf}(3,4,5,6,\cdots,2n) \\ & -\mathrm{pf}(1,3,5,6,\cdots,2n)\mathrm{pf}(2,4,5,6,\cdots,2n) \\ & +\mathrm{pf}(1,4,5,6,\cdots,2n)\mathrm{pf}(2,3,5,6,\cdots,2n), \end{aligned} \tag{2.34}$$

whose special cases are the Plücker relation and Jacobi's identity.

## § 2.5. Expansion fomulae of pfaffians and hafnians

We have the expansion formulae of pfaffians,

$$\begin{aligned} & \mathrm{pf}(x_1, x_2, 5, 6, \cdots, 2n) \\ & = \sum_{5 \le i < j \le 2n} (-1)^{i+j-1} \mathrm{pf}(x_1, x_2, i, j) \mathrm{pf}(5, 6, \cdots, \hat{i}, \cdots, \hat{j}, \cdots, 2n) \end{aligned} \tag{2.35}$$

and

$$\begin{aligned} & \mathrm{pf}(x_1, x_2, x_3, x_4, 5, 6, \cdots, 2n) \\ & = \sum_{5 \le i < j \le 2n} (-1)^{i+j-1} \mathrm{pf}(x_1, x_2, i, j) \mathrm{pf}(x_3, x_4, 5, 6, \cdots, \hat{i}, \cdots, \hat{j}, \cdots, 2n), \end{aligned} \tag{2.36}$$

where $\mathrm{pf}(x_j, x_k) = 0$ for $j, k = 1, 2, 3, 4$.

Similar expansion formulae hold for hafnians,

$$\begin{aligned} & \mathrm{hf}(x_1, x_2, 5, 6, \cdots, 2n) \\ & = \sum_{5 \le i < j \le 2n} \mathrm{hf}(x_1, x_2, i, j) \mathrm{hf}(5, 6, \cdots, \hat{i}, \cdots, \hat{j}, \cdots, 2n) \end{aligned} \tag{2.37}$$

and

$$
\begin{aligned}
&\mathrm{hf}(x_1, x_2, x_3, x_4, 5, 6, \cdots, 2n) \\
&\quad = \sum_{5\le i<j\le 2n} \mathrm{hf}(x_1, x_2, i, j)\mathrm{hf}(x_3, x_4, 5, 6, \cdots, \hat{\imath}, \cdots, \hat{\jmath}, \cdots, 2n), \\
&\quad = \sum_{5\le i<j\le 2n} \sum_{5\le k<l\le 2n} \mathrm{hf}(x_1, x_2, i, j)\mathrm{hf}(x_3, x_4, k, l) \\
&\qquad\qquad \times\mathrm{hf}(5, 6, \cdots, \hat{\imath}, \cdots, \hat{\jmath}, \cdots, \hat{k}, \cdots, \hat{l}, \cdots, 2n), \\
&\quad = \sum_{5\le i<j<k<l\le 2n} [\mathrm{hf}(x_1, x_2, i, j)\mathrm{hf}(x_3, x_4, k, l) + \mathrm{hf}(x_1, x_2, k, l)\mathrm{hf}(x_3, x_4, i, j) \\
&\qquad\qquad +\mathrm{hf}(x_1, x_2, i, k)\mathrm{hf}(x_3, x_4, j, l) + \mathrm{hf}(x_1, x_2, j, l)\mathrm{hf}(x_3, x_4, i, k) \\
&\qquad\qquad +\mathrm{hf}(x_1, x_2, i, l)\mathrm{hf}(x_3, x_4, j, k) + \mathrm{hf}(x_1, x_2, j, k)\mathrm{hf}(x_3, x_4, i, l)] \\
&\qquad\qquad \times\mathrm{hf}(5, 6, \cdots, \hat{\imath}, \cdots, \hat{\jmath}, \cdots, \hat{k}, \cdots, \hat{l}, \cdots, 2n), \qquad (2.38) \\
&\quad = \sum_{5\le i<j<k<l\le 2n} \mathrm{hf}(x_1, x_2, x_3, x_4, i, j, k, l) \\
&\qquad\qquad \times\mathrm{hf}(5, 6, \cdots, \hat{\imath}, \cdots, \hat{\jmath}, \cdots, \hat{k}, \cdots, \hat{l}, \cdots, 2n), \qquad (2.39)
\end{aligned}
$$

where $\mathrm{hf}(x_i, x_j) = 0$ for $i, j = 1, 2, 3, 4$.

Hereafter we write, for simplicity, $n$th-order, $(n-1)$th-oder and $(n-2)$th-order hafnians, respectively as

$$\mathrm{hf}(1, 2, \cdots, 2n) = \mathrm{hf}(\cdots)_n, \tag{2.40}$$

$$\mathrm{hf}(1, 2, \cdots, \hat{\jmath}, \cdots, \hat{k}, \cdots, 2n) = \mathrm{hf}(\hat{\jmath}, \hat{k})_n, \tag{2.41}$$

$$\mathrm{hf}(1, 2, \cdots, \hat{\imath}, \cdots, \hat{\jmath}, \cdots, \hat{k}, \cdots, \hat{l}, \cdots, 2n) = \mathrm{hf}(\hat{\imath}, \hat{\jmath}, \hat{k}, \hat{l})_n. \tag{2.42}$$

Let

$$\mathrm{hf}(\cdots)_{n-2} = \mathrm{hf}(5, 6, \cdots, 2n). \tag{2.43}$$

Then the expansion formulae of the hafnians are expressed by

$$\mathrm{hf}(x_1, x_2, 5, 6, \cdots, 2n) = \sum_{5\le i<j\le 2n} \mathrm{hf}(x_1, x_2, i, j)\mathrm{hf}(\hat{\imath}, \hat{\jmath})_{n-2} \tag{2.44}$$

and

$$
\begin{aligned}
&\mathrm{hf}(x_1, x_2, x_3, x_4, 5, 6, \cdots, 2n) \\
&\quad = \sum_{5\le i<j<k<l\le 2n} \mathrm{hf}(x_1, x_2, x_3, x_4, i, j, k, l)\mathrm{hf}(\hat{\imath}, \hat{\jmath}, \hat{k}, \hat{l})_{n-2}.
\end{aligned} \tag{2.45}
$$

A product of the hafnians is then expressed by

$$
\begin{aligned}
&\mathrm{hf}(x_1, x_2, 5, 6, \cdots, 2n)\mathrm{hf}(x_3, x_4, 5, 6, \cdots, 2n) \\
&\quad = \sum_{5\le i<j\le 2n} \sum_{5\le k<l\le 2n} \mathrm{hf}(x_1, x_2, i, j)\mathrm{hf}(x_3, x_4, k, l)\mathrm{hf}(\hat{\imath}, \hat{\jmath})_{n-2}\mathrm{hf}(\hat{k}, \hat{l})_{n-2}.
\end{aligned} \tag{2.46}
$$

Hereafter we omit the character hf denoting hafnians for short. Then the above expansion formula is written simply as

$$(2.47)\qquad \begin{aligned}&(x_1,x_2,5,6,\cdots,2n)(x_3,x_4,5,6,\cdots,2n)\\ &=\sum_{5\le i<j\le 2n}\sum_{5\le k<l\le 2n}(x_1,x_2,i,j)(x_3,x_4,k,l)(\hat{i},\hat{j})_{n-2}(\hat{k},\hat{l})_{n-2}.\end{aligned}$$

## § 3. Decomposition of products of the hafnians

Let us consider the following products of the hafnians,

$$(3.1)\qquad f_0=(1,2,3,4,5,\cdots,2n)(5,\cdots,2n),$$

$$(3.2)\qquad f_1=(1,2,5,\cdots,2n)(3,4,5,\cdots,2n),$$

$$(3.3)\qquad f_2=(1,3,5,\cdots,2n)(2,4,5,\cdots,2n),$$

$$(3.4)\qquad f_3=(1,4,5,\cdots,2n)(2,3,5,\cdots,2n).$$

We shall prove by induction that $f_0, f_1, f_2$ and $f_3$ are decomposed into the following forms,

$$(3.5)\qquad f_0=f_{01}+f_{02}+f_{03},$$

$$(3.6)\qquad f_1=f_{01}+f_{12}+f_{13},$$

$$(3.7)\qquad f_2=f_{02}+f_{12}+f_{23},$$

$$(3.8)\qquad f_3=f_{03}+f_{13}+f_{23}.$$

In the above expressions, if one expands $f_i$ and $f_j$ according to the definition(2.10), these two expressions possess common terms,which we denote $f_{i,j}$.

The proof consists of three steps. First we introduce new indices, $x_1, x_2, x_3, x_4$ in place of the indices $1,2,3,4$ and express $f_0, f_1, f_2$ and $f_3$ by the following forms,

$$(3.9)\qquad f_0=\overline{f}_1+\overline{f}_2+\overline{f}_3+h_0,$$

$$(3.10)\qquad f_1=\overline{f}_1+g_1,$$

$$(3.11)\qquad f_2=\overline{f}_2+g_2,$$

$$(3.12)\qquad f_3=\overline{f}_3+g_3,$$

where $\overline{f}_j$ is defined by Eqs.(3.67), (3.69) and (3.71) for $j=1,2,3$ respectively, and

$$(3.13)\qquad h_0=(x_1,x_2,x_3,x_4,5,\cdots,2n)(5,\cdots,2n),$$

$$(3.14)\qquad g_1=(x_1,x_2,5,\cdots,2n)(x_3,x_4,5,\cdots,2n),$$

$$(3.15)\qquad g_2=(x_1,x_3,5,\cdots,2n)(x_2,x_4,5,\cdots,2n),$$

$$(3.16)\qquad g_3=(x_1,x_4,5,\cdots,2n)(x_2,x_3,5,\cdots,2n),$$

and $(x_i, x_j) = 0$ for $i, j = 1, 2, 3, 4$.

Second we introduce new indices,$y_1, y_2, y_3, y_4$ in place of the indices $x_1, x_2, x_3, x_4$ and express $g_1, g_2$ and $g_3$ by the following forms,

$$g_1 = \overline{g}_{12} + \overline{g}_{13} + h_1, \tag{3.17}$$

$$g_2 = \overline{g}_{12} + \overline{g}_{23} + h_2, \tag{3.18}$$

$$g_3 = \overline{g}_{23} + \overline{g}_{23} + h_3, \tag{3.19}$$

where

$$h_1 = (y_1, y_2, 5, \cdots, 2n)(y_3, y_4, 5, \cdots, 2n), \tag{3.20}$$

$$h_2 = (y_1, y_3, 5, \cdots, 2n)(y_2, y_4, 5, \cdots, 2n), \tag{3.21}$$

$$h_3 = (y_1, y_4, 5, \cdots, 2n)(y_2, y_3, 5, \cdots, 2n), \tag{3.22}$$

where $(y_i, y_j) = 0$ for $i, j = 1, 2, 3, 4$ and

$$(y_i, j) = a_{i,j} \quad \text{and} \quad a_{i,k}a_{j,k} = 0, \tag{3.23}$$

for $i, j = 1, 2, 3, 4$ and for $k = 4, 5, \cdots, 2n$.

Finally we prove, by induction, decomposition of $h_0, h_1, h_2$ and $h_3$, which gives immediately the decomposition of $f_0, f_1, f_2$ and $f_3$.

## § 3.1. Decomposition of products of $(n-2)$th-oder hafnians

In order to use induction we consider the decomposition of $f_0, f_1, f_2$ and $f_3$ for $(n-2)$th-oder hafnians such as

$$\begin{aligned} f_0^{n-2} &= (5, 6, 7, 8, 9, 10, \cdots, 2n)(9, 10, \cdots, 2n), \\ &= f_{01}^{n-2} + f_{02}^{n-2} + f_{03}^{n-2}. \end{aligned} \tag{3.24}$$

$$\begin{aligned} f_1^{n-2} &= (5, 6, 9, 10, \cdots, 2n)(7, 8, 9, 10, \cdots, 2n), \\ &= f_{01}^{n-2} + f_{12}^{n-2} + f_{13}^{n-2}. \end{aligned} \tag{3.25}$$

$$\begin{aligned} f_2^{n-2} &= (5, 7, 9, 10, \cdots, 2n)(6, 8, 9, 10, \cdots, 2n), \\ &= f_{02}^{n-2} + f_{12}^{n-2} + f_{23}^{n-2}. \end{aligned} \tag{3.26}$$

$$\begin{aligned} f_3^{n-2} &= (5, 8, 9, 10, \cdots, 2n)(6, 7, 9, 10, \cdots, 2n), \\ &= f_{03}^{n-2} + f_{13}^{n-2} + f_{23}^{n-2}, \end{aligned} \tag{3.27}$$

which may be written, using

$$(\cdots)_{n-2} = (5, 6, 7, 8, 9, 10, \cdots, 2n), \tag{3.28}$$

$$(\hat{5}, \hat{6}, \hat{7}, \hat{8})_{n-2} = (9, 10, \cdots, 2n), \tag{3.29}$$

$$f_{ij}(\hat{5}, \hat{6}, \hat{7}, \hat{8}) = f_{ij}^{n-2}, \text{ for } i, j = 0, 1, 2, 3, \tag{3.30}$$

as

$$
\begin{aligned}
f_0^{n-2} &= (\hat{5},\hat{6},\hat{7},\hat{8})_{n-2}(\cdots)_{n-2}, \\
&= f_{01}(\hat{5},\hat{6},\hat{7},\hat{8}) + f_{02}(\hat{5},\hat{6},\hat{7},\hat{8}) + f_{03}(\hat{5},\hat{6},\hat{7},\hat{8}), \qquad (3.31)\\
f_1^{n-2} &= (\hat{5},\hat{6})_{n-2}(\hat{7},\hat{8})_{n-2}, \\
&= f_{01}(\hat{5},\hat{6},\hat{7},\hat{8}) + f_{12}(\hat{5},\hat{6},\hat{7},\hat{8}) + f_{13}(\hat{5},\hat{6},\hat{7},\hat{8}), \qquad (3.32)\\
f_2^{n-2} &= (\hat{5},\hat{7})_{n-2}(\hat{6},\hat{8})_{n-2}, \\
&= f_{02}(\hat{5},\hat{6},\hat{7},\hat{8}) + f_{12}(\hat{5},\hat{6},\hat{7},\hat{8}) + f_{23}(\hat{5},\hat{6},\hat{7},\hat{8}), \qquad (3.33)\\
f_3^{n-2} &= (\hat{5},\hat{8})_{n-2}(\hat{6},\hat{7})_{n-2}, \\
&= f_{03}(\hat{5},\hat{8},\hat{6},\hat{7}) + f_{13}(\hat{5},\hat{8},\hat{6},\hat{7}) + f_{23}(\hat{5},\hat{8},\hat{6},\hat{7}). \qquad (3.34)
\end{aligned}
$$

In general the decomposition of products of $(n-2)$th-oder hafnians are expressed by

$$
\begin{aligned}
f_0^{n-2} &= (\hat{i},\hat{j},\hat{k},\hat{l})_{n-2}(\cdots)_{n-2}, \\
&= f_{01}(\hat{i},\hat{j},\hat{k},\hat{l}) + f_{02}(\hat{i},\hat{j},\hat{k},\hat{l}) + f_{03}(\hat{i},\hat{j},\hat{k},\hat{l}), \qquad (3.35)\\
f_1^{n-2} &= (\hat{i},\hat{j})_{n-2}(\hat{k},\hat{l})_{n-2}, \\
&= f_{01}(\hat{i},\hat{j},\hat{k},\hat{l}) + f_{12}(\hat{i},\hat{j},\hat{k},\hat{l}) + f_{13}(\hat{i},\hat{j},\hat{k},\hat{l}), \qquad (3.36)\\
f_2^{n-2} &= (\hat{i},\hat{k})_{n-2}(\hat{j},\hat{l})_{n-2}, \\
&= f_{02}(\hat{i},\hat{j},\hat{k},\hat{l}) + f_{12}(\hat{i},\hat{j},\hat{k},\hat{l}) + f_{23}(\hat{i},\hat{j},\hat{k},\hat{l}), \qquad (3.37)\\
f_3^{n-2} &= (\hat{i},\hat{l})_{n-2}(\hat{j},\hat{k})_{n-2}, \\
&= f_{03}(\hat{i},\hat{l},\hat{j},\hat{k}) + f_{13}(\hat{i},\hat{l},\hat{j},\hat{k}) + f_{23}(\hat{i},\hat{l},\hat{j},\hat{k}), \qquad (3.38)
\end{aligned}
$$

where indices $i,j,k,l$ are members of a list $\{9,10,\cdots,2n\}$.

We assume the decomposition of products of $(n-2)$th-order hafnians of the forms,(3.35), (3.36), (3.37) and (3.38) for the use of induction.

## § 3.2. Expressions for $f_0, f_1, f_2$ and $f_3$

Consider an entry $(j,k)$ of the hafnian $(1,2,\cdots,2n)$. Let an index $j$ be a sum of indices $\bar{j}$ and $x_j$ for $j=1,2,3,4$,namely

$$j = \bar{j} + x_j, \quad \text{for } j = 1,2,3,4 \qquad (3.39)$$

and introduce new entries of the hafnians,

$$(\bar{j},\bar{k}) = a_{jk}, \tag{3.40}$$

$$(x_j, x_k) = 0, \tag{3.41}$$

$$(x_j, \bar{k}) = 0, \quad \text{for } j,k = 1,2,3,4, \tag{3.42}$$

$$(\bar{j}, k) = 0, \tag{3.43}$$

$$(x_j, k) = a_{jk}, \quad \text{for } j = 1,2,3,4 \text{ and } k = 5,6,\cdots,2n, \tag{3.44}$$

$$(j,k) = a_{jk}, \quad \text{for } j,k = 5,6,\cdots,2n, \tag{3.45}$$

so that

$$(j,k) = (\bar{j}+x_j, \bar{k}+x_k) = (\bar{j},\bar{k}) + (\bar{j},x_k) + (x_j,\bar{k}) + (x_j,x_k) \tag{3.46}$$

$$= a_{j,k}, \quad \text{for } j,k = 1,2,3,4, \tag{3.47}$$

and

$$(j,k) = (\bar{j}+x_j, k) = (\bar{j},k) + (x_j,k) \tag{3.48}$$

$$= a_{jk}, \quad \text{for } j = 1,2,3,4 \text{ and } k = 5,6,\cdots,2n. \tag{3.49}$$

Accordingly we have, for example,

$$(1,2,5,\cdots,2n) = (\bar{1}+x_1, \bar{2}+x_2, 5,6,\cdots,2n) \tag{3.50}$$

$$= (\bar{1},\bar{2},5,6,\cdots,2n) + (\bar{1},x_2,5,6,\cdots,2n) \tag{3.51}$$

$$+(x_1,\bar{2},5,6,\cdots,2n) + (x_1,x_2,5,6,\cdots,2n) \tag{3.52}$$

$$= a_{12}(5,6,\cdots,2n) + (x_1,x_2,5,6,\cdots,2n), \tag{3.53}$$

Then hafnians of order $(n-1)$ and of order $n$ are written as sums of terms of special form, respectively, as follows

$$(1,2,5,\cdots,2n) = a_{12}(5,\cdots,2n) + (x_1,x_2,5,\cdots,2n), \tag{3.54}$$

$$(3,4,5,\cdots,2n) = a_{34}(5,\cdots,2n) + (x_3,x_4,5,\cdots,2n), \tag{3.55}$$

$$(1,3,5,\cdots,2n) = a_{13}(5,\cdots,2n) + (x_1,x_3,5,\cdots,2n), \tag{3.56}$$

$$(2,4,5,\cdots,2n) = a_{24}(5,\cdots,2n) + (x_2,x_4,5,\cdots,2n), \tag{3.57}$$

$$(1,4,5,\cdots,2n) = a_{14}(5,\cdots,2n) + (x_1,x_4,5,\cdots,2n), \tag{3.58}$$

$$(2,3,5,\cdots,2n) = a_{23}(5,\cdots,2n) + (x_2,x_3,5,\cdots,2n), \tag{3.59}$$

$$\begin{aligned}(1,2,3,4,5,\cdots,2n) &= (a_{12}a_{34} + a_{13}a_{24} + a_{14}a_{23})(5,6,\cdots,2n)\\ &+a_{12}(x_3,x_4,5,\cdots,2n) + a_{34}(x_1,x_2,5,\cdots,2n)\\ &+a_{13}(x_2,x_4,5,\cdots,2n) + a_{24}(x_1,x_3,5,\cdots,2n)\\ &+a_{14}(x_2,x_3,5,\cdots,2n) + a_{23}(x_1,x_4,5,\cdots,2n)\\ &+(x_1,x_2,x_3,x_4,5,\cdots,2n).\end{aligned} \tag{3.60}$$

Then $f_0, f_1, f_2, f_3$ are written, using the above relations, as

$$f_0 = \overline{f}_0 + h_0, \tag{3.61}$$

$$f_1 = \overline{f}_1 + g_1, \tag{3.62}$$

$$f_2 = \overline{f}_2 + g_2, \tag{3.63}$$

$$f_3 = \overline{f}_3 + g_3, \tag{3.64}$$

where we have used $h_0$ instead of $g_0$ for later convenience, and

$$\begin{aligned}\overline{f}_0 = [&(a_{12}a_{34} + a_{13}a_{24} + a_{14}a_{23})(5, 6, \cdots, 2n) \\ &+a_{12}(x_3, x_4, 5, \cdots, 2n) + a_{34}(x_1, x_2, 5, \cdots, 2n) \\ &+a_{13}(x_2, x_4, 5, \cdots, 2n) + a_{24}(x_1, x_3, 5, \cdots, 2n) \\ &+a_{14}(x_2, x_3, 5, \cdots, 2n) + a_{23}(x_1, x_4, 5, \cdots, 2n)] \\ &\times(5, 6, \cdots, 2n),\end{aligned} \tag{3.65}$$

$$h_0 = (x_1, x_2, x_3, x_4, 5, \cdots, 2n)(5, \cdots, 2n), \tag{3.66}$$

$$\begin{aligned}\overline{f}_1 = [&a_{12}a_{34}(5, 6, \cdots, 2n) \\ &+a_{12}(x_3, x_4, 5, \cdots, 2n) + a_{34}(x_1, x_2, 5, \cdots, 2n)] \\ &\times(5, 6, \cdots, 2n),\end{aligned} \tag{3.67}$$

$$g_1 = (x_1, x_2, 5, \cdots, 2n)(x_3, x_4, 5, \cdots, 2n), \tag{3.68}$$

$$\begin{aligned}\overline{f}_2 = [&a_{13}a_{24}(5, 6, \cdots, 2n) \\ &+a_{13}(x_2, x_4, 5, \cdots, 2n) + a_{24}(x_1, x_3, 5, \cdots, 2n)] \\ &\times(5, 6, \cdots, 2n),\end{aligned} \tag{3.69}$$

$$g_2 = (x_1, x_3, 5, \cdots, 2n)(x_2, x_4, 5, \cdots, 2n), \tag{3.70}$$

$$\begin{aligned}\overline{f}_3 = [&a_{14}a_{23}(5, 6, \cdots, 2n) \\ &+a_{14}(x_2, x_3, 5, \cdots, 2n) + a_{23}(x_1, x_4, 5, \cdots, 2n)] \\ &\times(5, 6, \cdots, 2n),\end{aligned} \tag{3.71}$$

$$g_3 = (x_1, x_4, 5, \cdots, 2n)(x_2, x_3, 5, \cdots, 2n). \tag{3.72}$$

Then we find a decomposition of $\overline{f}_0$,

$$\overline{f}_0 = \overline{f}_1 + \overline{f}_2 + \overline{f}_3. \tag{3.73}$$

Accordingly $f_0, f_1, f_2, f_3$ are expressed by

$$f_0 = \overline{f}_1 + \overline{f}_2 + \overline{f}_3 + h_0, \tag{3.74}$$

$$f_1 = \overline{f}_1 + g_1, \tag{3.75}$$

$$f_2 = \overline{f}_2 + g_2, \tag{3.76}$$

$$f_3 = \overline{f}_3 + g_3. \tag{3.77}$$

### § 3.3. Expressions for $g_1, g_2$ and $g_3$

Let $\delta_{i,j}$ be Kronecker's delta function,

$$\delta_{i,j} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} . \tag{3.78}$$

We have an identity, for $i < j$ and $k < l$,

$$\begin{aligned} 1 = &\ \delta_{i,k}\delta_{j,l} + (\delta_{i,k} + \delta_{i,l} + \delta_{j,k} + \delta_{j,l})(1 - \delta_{i,k}\delta_{j,l}) \\ &+(1 - \delta_{i,k})(1 - \delta_{i,l})(1 - \delta_{j,k})(1 - \delta_{j,l}). \end{aligned} \tag{3.79}$$

Accordingly a sum of a product of arbitrary functions, $v(i, j)$ and $w(k, l)$ are written as

$$\begin{aligned} \sum_{i<j}\sum_{k<l} v(i, j)w(k, l) &= \sum_{i<j}\sum_{k<l} \delta_{i,k}\delta_{j,l} v(i, j)w(k, l) \\ &+ \sum_{i<j}\sum_{k<l} (\delta_{i,k} + \delta_{i,l} + \delta_{j,k} + \delta_{j,l})(1 - \delta_{i,k}\delta_{j,l}) v(i, j)w(k, l) \\ &+ \sum_{i<j}\sum_{k<l} (1 - \delta_{i,k})(1 - \delta_{i,l})(1 - \delta_{j,k})(1 - \delta_{j,l}) v(i, j)w(k, l), \end{aligned} \tag{3.80}$$

which is rewritten as

$$\begin{aligned} \sum_{i<j}\sum_{k<l} v(i, j)w(k, l) &= \sum_{i<j} v(i, j)w(i, j) + \sum_{i<j<k} (v(i, j)w(i, k) + v(i, k)w(i, j)) \\ &+ \sum_{i<k<j} (v(i, j)w(k, j) + v(k, j)w(i, j)) + \sum_{i<j<k} (v(i, j)w(j, k) + v(j, k)w(i, j)) \\ &+ \sum_{i<j<k<l} (v(i, j)w(k, l) + v(i, k)w(j, l) + v(i, l)w(j, k) \\ &\quad +v(k, l)w(i, j) + v(j, l)w(i, k) + v(j, k)w(i, l)). \end{aligned} \tag{3.81}$$

We expand $g_1$,

$$g_1 = (x_1, x_2, 5, 6, \cdots, 2n)(x_3, x_4, 5, 6, \cdots, 2n) \tag{3.82}$$

$$= \sum_{5 \le i<j \le 2n} \sum_{5 \le k<l \le 2n} (x_1, x_2, i, j)(x_3, x_4, k, l)(\hat{i}, \hat{j})_{n-2}(\hat{k}, \hat{l})_{n-2}. \tag{3.83}$$

Let

$$u(i, j) = (x_1, x_2, i, j)(\hat{i}, \hat{j}), \quad v(i, j) = (x_3, x_4, i, j)(\hat{i}, \hat{j}). \tag{3.84}$$

We obtain, using (3.81),

$$
\begin{aligned}
g_1 = &\sum_{5\le i<j\le 2n}(x_1,x_2,i,j)(x_3,x_4,i,j)[(\hat{i},\hat{j})_{n-2}]^2\\
&+\sum_{5\le i<j<k\le 2n}[(x_1,x_2,i,j)(x_3,x_4,i,k)+(x_1,x_2,i,k)(x_3,x_4,i,j)]\\
&\qquad\qquad\times(\hat{i},\hat{j})_{n-2}(\hat{i},\hat{k})_{n-2}\\
&+\sum_{5\le i<k<j\le 2n}[(x_1,x_2,i,j)(x_3,x_4,k,j)+(x_1,x_2,k,j)(x_3,x_4,i,j)]\\
&\qquad\qquad\times(\hat{i},\hat{j})_{n-2}(\hat{k},\hat{j})_{n-2}\\
&+\sum_{5\le i<j<k\le 2n}[(x_1,x_2,i,j)(x_3,x_4,j,k)+(x_1,x_2,j,k)(x_3,x_4,i,j)]\\
&\qquad\qquad\times(\hat{i},\hat{j})_{n-2}(\hat{j},\hat{k})_{n-2}\\
&+\sum_{5\le i<j<k<l\le 2n}[(x_1,x_2,i,j)(x_3,x_4,k,l)+(x_1,x_2,k,l)(x_3,x_4,i,j)]\\
&\qquad\qquad\times(\hat{i},\hat{j})_{n-2}(\hat{k},\hat{l})_{n-2}.\\
&\qquad\qquad+[(x_1,x_2,i,k)(x_3,x_4,j,l)+(x_1,x_2,j,l)(x_3,x_4,i,k)]\\
&\qquad\qquad\times(\hat{i},\hat{k})_{n-2}(\hat{j},\hat{l})_{n-2}.\\
&\qquad\qquad+[(x_1,x_2,i,l)(x_3,x_4,j,k)+(x_1,x_2,j,k)(x_3,x_4,i,l)]\\
&\qquad\qquad\times(\hat{i},\hat{l})_{n-2}(\hat{j},\hat{k})_{n-2}.
\end{aligned}\tag{3.85}
$$

On the other hand we have the following expressions,

$$
\begin{aligned}
&(x_1,x_2,i,j)(x_3,x_4,i,j)\\
&\quad=a_{1i}a_{3i}a_{2j}a_{4j}+a_{1j}a_{3j}a_{2i}a_{4i}+a_{1i}a_{4i}a_{2j}a_{3j}+a_{1j}a_{4j}a_{2i}a_{3i},\\
&\qquad\qquad\text{for } 5\le i<j\le 2n,
\end{aligned}\tag{3.86}
$$

$$
\begin{aligned}
&(x_1,x_2,i,j)(x_3,x_4,i,k)+(x_1,x_2,i,k)(x_3,x_4,i,j)\\
&\quad=a_{1i}a_{3i}(x_2,x_4,j,k)+a_{2i}a_{4i}(x_1,x_3,j,k)\\
&\qquad+a_{1i}a_{4i}(x_2,x_3,j,k)+a_{2i}a_{3i}(x_1,x_4,j,k),
\end{aligned}\tag{3.87}
$$

$$
\begin{aligned}
&(x_1,x_2,i,j)(x_3,x_4,k,j)+(x_1,x_2,k,j)(x_3,x_4,i,j)\\
&\quad=a_{1j}a_{3j}(x_2,x_4,i,k)+a_{2j}a_{4j}(x_1,x_3,i,k)\\
&\qquad+a_{1j}a_{4j}(x_2,x_3,i,k)+a_{2j}a_{3j}(x_1,x_4,i,k),
\end{aligned}\tag{3.88}
$$

$$
\begin{aligned}
&(x_1,x_2,i,j)(x_3,x_4,j,k)+(x_1,x_2,j,k)(x_3,x_4,i,j)\\
&\quad=a_{1j}a_{3j}(x_2,x_4,i,k)+a_{2j}a_{4j}(x_1,x_3,i,k)\\
&\qquad+a_{1j}a_{4j}(x_2,x_3,i,k)+a_{2j}a_{3j}(x_1,x_4,i,k),\\
&\qquad\qquad\text{for } 5\le i<j<k\le 2n,
\end{aligned}\tag{3.89}
$$

which show that all terms in Eq.(3.85) except the last one vanish provided that

$$a_{ik}a_{jk} = 0 \quad \text{for } i,j = 1,2,3,4 \text{ and for } k = 5,6,\cdots,2n. \tag{3.90}$$

We introduce new indices $y_i$ for $i = 1,2,3,4$ in place of $x_i$.

Let

$$(y_i, y_j) = 0, \tag{3.91}$$

and

$$(y_i, j) = a_{i,j} \quad \text{and} \quad a_{i,k}a_{j,k} = 0, \tag{3.92}$$

for $i,j = 1,2,3,4$ and for $k = 4,5,\cdots,2n$.

Accordingly we write $g_1$ as

$$g_1 = \overline{g}_1 + h_1, \tag{3.93}$$

where

$$\begin{aligned}
\overline{g}_1 = &\sum_{5\le i<j\le 2n} [a_{1i}a_{3i}a_{2j}a_{4j} + a_{1j}a_{3j}a_{2i}a_{4i} + a_{1i}a_{4i}a_{2j}a_{3j} + a_{1j}a_{4j}a_{2i}a_{3i}][(\hat{i},\hat{j})_{n-2}]^2 \\
&+ \sum_{5\le i<j<k\le 2n} [a_{1i}a_{3i}(x_2,x_4,j,k) + a_{2i}a_{4i}(x_1,x_3,j,k) \\
&\qquad + a_{1i}a_{4i}(x_2,x_3,j,k) + a_{2i}a_{3i}(x_1,x_4,j,k)](\hat{i},\hat{j})_{n-2}(\hat{i},\hat{k})_{n-2} \\
&+ \sum_{5\le i<k<j\le 2n} [a_{1j}a_{3j}(x_2,x_4,i,k) + a_{2j}a_{4j}(x_1,x_3,i,k) \\
&\qquad + a_{1j}a_{4j}(x_2,x_3,i,k) + a_{2j}a_{3j}(x_1,x_4,i,k)](\hat{i},\hat{j})_{n-2}(\hat{k},\hat{j})_{n-2} \\
&+ \sum_{5\le i<j<k\le 2n} [a_{1j}a_{3j}(x_2,x_4,i,k) + a_{2j}a_{4j}(x_1,x_3,i,k) \\
&\qquad + a_{1j}a_{4j}(x_2,x_3,i,k) + a_{2j}a_{3j}(x_1,x_4,i,k)](\hat{i},\hat{j})_{n-2}(\hat{j},\hat{k})_{n-2}
\end{aligned} \tag{3.94}$$

and

$$h_1 = (y_1,y_2,5,6,\cdots,2n)(y_3,y_4,5,6,\cdots,2n), \tag{3.95}$$

using the condition(3.92).

Interchanging the index 2 with the index 3 in Eq.(3.94) we obtain

$$g_2 = (x_1,x_3,5,6,\cdots,2n)(x_2,x_4,5,6,\cdots,2n) = \overline{g}_2 + h_2, \tag{3.96}$$

where

$$\begin{aligned}
\overline{g}_2 &= \overline{g}_1|_{2\rightleftharpoons 3},\\
&= \sum_{5\le i<j\le 2n} [a_{1i}a_{2i}a_{3j}a_{4j} + a_{1j}a_{2j}a_{3i}a_{4i} + a_{1i}a_{4i}a_{2j}a_{3j} + a_{1j}a_{4j}a_{2i}a_{3i}][(\hat{i},\hat{j})_{n-2}]^2\\
&+ \sum_{5\le i<j<k\le 2n} [a_{1i}a_{2i}(x_3,x_4,j,k) + a_{3i}a_{4i}(x_1,x_3,j,k)\\
&\qquad +a_{1i}a_{4i}(x_2,x_3,j,k) + a_{2i}a_{3i}(x_1,x_4,j,k)](\hat{i},\hat{j})_{n-2}(\hat{i},\hat{k})_{n-2}\\
&+ \sum_{5\le i<k<j\le 2n} [a_{1j}a_{2j}(x_3,x_4,i,k) + a_{3j}a_{4j}(x_1,x_2,i,k)\\
&\qquad +a_{1j}a_{4j}(x_2,x_3,i,k) + a_{2j}a_{3j}(x_1,x_4,i,k)](\hat{i},\hat{j})_{n-2}(\hat{k},\hat{j})_{n-2}\\
&+ \sum_{5\le i<j<k\le 2n} [a_{1j}a_{2j}(x_3,x_4,i,k) + a_{3j}a_{4j}(x_1,x_2,i,k)\\
&\qquad +a_{1j}a_{4j}(x_2,x_3,i,k) + a_{2j}a_{3j}(x_1,x_4,i,k)](\hat{i},\hat{j})_{n-2}(\hat{j},\hat{k})_{,n-2}
\end{aligned} \tag{3.97}$$

and

$$h_2 = (y_1,y_3,5,6,\cdots,2n)(y_2,y_4,5,6,\cdots,2n). \tag{3.98}$$

Similarly interchanging the index 2 with the index 4 in Eq.(3.94) we obtain

$$g_3 = (x_1,x_4,5,6,\cdots,2n)(x_2,x_3,5,6,\cdots,2n) = \overline{g}_3 + h_3, \tag{3.99}$$

where

$$\begin{aligned}
\overline{g}_3 &= \overline{g}_1|_{2\rightleftharpoons 4},\\
&= \sum_{5\le i<j\le 2n} [a_{1i}a_{3i}a_{2j}a_{4j} + a_{1j}a_{3j}a_{2i}a_{4i} + a_{1i}a_{2i}a_{3j}a_{4j} + a_{1j}a_{2j}a_{3i}a_{4i}][(\hat{i},\hat{j})_{n-2}]^2\\
&+ \sum_{5\le i<j<k\le 2n} [a_{1i}a_{3i}(x_2,x_4,j,k) + a_{2i}a_{4i}(x_1,x_3,j,k)\\
&\qquad +a_{1i}a_{2i}(x_3,x_4,j,k) + a_{3i}a_{4i}(x_1,x_2,j,k)](\hat{i},\hat{j})_{n-2}(\hat{i},\hat{k})_{n-2}\\
&+ \sum_{5\le i<k<j\le 2n} [a_{1j}a_{3j}(x_2,x_4,i,k) + a_{2j}a_{4j}(x_1,x_3,i,k)\\
&\qquad +a_{1j}a_{2j}(x_3,x_4,i,k) + a_{3j}a_{4j}(x_1,x_2,i,k)](\hat{i},\hat{j})_{n-2}(\hat{k},\hat{j})_{n-2}\\
&+ \sum_{5\le i<j<k\le 2n} [a_{1j}a_{3j}(x_2,x_4,i,k) + a_{2j}a_{4j}(x_1,x_3,i,k)\\
&\qquad +a_{1j}a_{2j}(x_3,x_4,i,k) + a_{3j}a_{4j}(x_1,x_2,i,k)](\hat{i},\hat{j})_{n-2}(\hat{j},\hat{k})_{n-2}
\end{aligned} \tag{3.100}$$

and

$$h_3 = (y_1,y_4,5,6,\cdots,2n)(y_2,y_3,5,6,\cdots,2n). \tag{3.101}$$

From the expressions of $\overline{g}_1, \overline{g}_2$ and $\overline{g}_3$ we find the following decomposition,

$$\overline{g}_1 = \overline{g}_{12} + \overline{g}_{13}, \tag{3.102}$$

$$\overline{g}_2 = \overline{g}_{12} + \overline{g}_{23}, \tag{3.103}$$

$$\overline{g}_3 = \overline{g}_{13} + \overline{g}_{23}, \tag{3.104}$$

where

$$\begin{aligned}\overline{g}_{12} = &\sum_{5\le i<j\le 2n} [a_{1i}a_{4i}a_{2j}a_{3j} + a_{1j}a_{4j}a_{2i}a_{3i}][(\hat{i},\hat{j})_{n-2}]^2 \\ &+ \sum_{5\le i<j<k\le 2n} [a_{1i}a_{4i}(x_2,x_3,j,k) + a_{2i}a_{3i}(x_1,x_4,j,k)](\hat{i},\hat{j})_{n-2}(\hat{i},\hat{k})_{n-2} \\ &+ \sum_{5\le i<k<j\le 2n} [a_{1j}a_{4j}(x_2,x_3,i,k) + a_{2j}a_{3j}(x_1,x_4,i,k)](\hat{i},\hat{j})_{n-2}(\hat{k},\hat{j})_{n-2} \\ &+ \sum_{5\le i<j<k\le 2n} [a_{1j}a_{4j}(x_2,x_3,i,k) + a_{2j}a_{3j}(x_1,x_4,i,k)](\hat{i},\hat{j})_{n-2}(\hat{j},\hat{k})_{n-2},\end{aligned} \tag{3.105}$$

$$g_{13} = \overline{g}_{12}|_{3\rightleftharpoons 4}, \tag{3.106}$$

$$g_{23} = \overline{g}_{12}|_{2\rightleftharpoons 4}. \tag{3.107}$$

Accordingly $g_1, g_2, g_3$ are decomposed into

$$g_1 = \overline{g}_{12} + \overline{g}_{13} + h_1, \tag{3.108}$$

$$g_2 = \overline{g}_{12} + \overline{g}_{23} + h_2, \tag{3.109}$$

$$g_3 = \overline{g}_{23} + \overline{g}_{23} + h_3. \tag{3.110}$$

## § 3.4. Expressions for $h_0, h_1, h_2$ and $h_3$

We have $h_0$ defined by Eq.(3.66),

$$h_0 = (y_1, y_2, y_3, y_4, 5, 6, \cdots, 2n)(5, 6, \cdots, 2n), \tag{3.111}$$

which is expanded, using Eq.(2.39), in

$$= \sum_{5\le i<j<k<l\le 2n} (y_1, y_2, y_3, y_4, i, j, k, l)(\hat{i},\hat{j},\hat{k},\hat{l})_{n-2}(\cdots)_{n-2}. \tag{3.112}$$

Here we assume that the product $(\hat{i},\hat{j},\hat{k},\hat{l})_{n-2}(\cdots)_{n-2}$ is decomposed into

$$(\hat{i},\hat{j},\hat{k},\hat{l})_{n-2}(\cdots)_{n-2} = f_{01}(\hat{i},\hat{j},\hat{k},\hat{l}) + f_{02}(\hat{i},\hat{j},\hat{k},\hat{l}) + f_{03}(\hat{i},\hat{j},\hat{k},\hat{l}). \tag{3.113}$$

The hafnian $(y_1, y_2, y_3, y_4, i, j, k, l)$ in Eq.(3.112) is expanded in three different ways,

$$
\begin{aligned}
&(y_1, y_2, y_3, y_4, i, j, k, l)\\
&= (y_1, y_2, i, j)(y_3, y_4, k, l) + (y_1, y_2, k, l)(y_3, y_4, i, j)\\
&+(y_1, y_2, i, k)(y_3, y_4, j, l) + (y_1, y_2, j, l)(y_3, y_4, i, k)\\
&+(y_1, y_2, i, l)(y_3, y_4, j, k) + (y_1, y_2, j, k)(y_3, y_4, i, i) \qquad (3.114)\\
&= (y_1, y_3, i, j)(y_2, y_4, k, l) + (y_1, y_3, k, l)(y_2, y_4, i, j)\\
&+(y_1, y_3, i, k)(y_2, y_4, j, l) + (y_1, y_3, j, l)(y_2, y_4, i, k)\\
&+(y_1, y_3, i, l)(y_2, y_4, j, k) + (y_1, y_3, j, k)(y_2, y_4, i, i) \qquad (3.115)\\
&= (y_1, y_4, i, j)(y_2, y_3, k, l) + (y_1, y_4, k, l)(y_2, y_3, i, j)\\
&+(y_1, y_4, i, k)(y_2, y_3, j, l) + (y_1, y_4, j, l)(y_2, y_3, i, k)\\
&+(y_1, y_4, i, l)(y_2, y_3, j, k) + (y_1, y_4, j, k)(y_2, y_3, i, i). \qquad (3.116)
\end{aligned}
$$

Substituting Eqs.(3.114) and (3.113) into Eq.(3.112) we obtain an expression for $h_0$,

$$
\begin{aligned}
h_0 = \sum_{5\le i<j<k<l\le 2n} &[(y_1, y_2, i, j)(y_3, y_4, k, l) + (y_1, y_2, k, l)(y_3, y_4, i, j)\\
&+(y_1, y_2, i, k)(y_3, y_4, j, l) + (y_1, y_2, j, l)(y_3, y_4, i, k)\\
&+(y_1, y_2, i, l)(y_3, y_4, j, k) + (y_1, y_2, i, l)(y_3, y_4, j, k)]\\
&\times[f_{01}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{02}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{03}(\hat{i}, \hat{j}, \hat{k}, \hat{l})]. \qquad (3.117)
\end{aligned}
$$

On the other hand,replacing $x_i$ in Eq.(3.85) by $y_i$ for $i = 1, 2, 3, 4$ and using Eqs.(3.92) we obtain

$$
\begin{aligned}
h_1 &= (y_1, y_2, 5, 6, \cdots, 2n)(y_3, y_4, 5, 6, \cdots, 2n),\\
&= \sum_{5\le i<j<k<l\le 2n} [(y_1, y_2, i, j)(y_3, y_4, k, l) + (y_1, y_2, k, l)(y_3, y_4, i, j)](\hat{i}, \hat{j})_{n-2}(\hat{k}, \hat{l})_{n-2}\\
&\quad +[(y_1, y_2, i, k)(y_3, y_4, j, l) + (y_1, y_2, j, l)(y_3, y_4, i, k)](\hat{i}, \hat{k})_{n-2}(\hat{j}, \hat{l})_{n-2}\\
&\quad +[(y_1, y_2, i, l)(y_3, y_4, j, k) + (y_1, y_2, j, k)(y_3, y_4, i, l)](\hat{i}, \hat{l})_{n-2}(\hat{j}, \hat{k})_{n-2}.
\end{aligned}
\qquad (3.118)
$$

We assume that the products of the $(n-2)$th-order hafnians are decomposed into

$$(\hat{i}, \hat{j})_{n-2}(\hat{k}, \hat{l})_{n-2} = f_{01}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{12}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{13}(\hat{i}, \hat{j}, \hat{k}, \hat{l}), \qquad (3.119)$$

$$(\hat{i}, \hat{k})_{n-2}(\hat{j}, \hat{l})_{n-2} = f_{02}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{12}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{23}(\hat{i}, \hat{j}, \hat{k}, \hat{l}), \qquad (3.120)$$

$$(\hat{i}, \hat{l})_{n-2}(\hat{j}, \hat{k})_{n-2} = f_{03}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{13}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{23}(\hat{i}, \hat{j}, \hat{k}, \hat{l}). \qquad (3.121)$$

Substituting Eqs.(3.119),(3.120) and (3.121) into Eq.(3.118) we obtain an expression for

$h_1$,

$$\begin{aligned} h_1 = \sum_{5\le i<j<k<l\le 2n} &[(y_1,y_2,i,j)(y_3,y_4,k,l)+(y_1,y_2,k,l)(y_3,y_4,i,j)] \\ &\times[f_{01}(\hat{i},\hat{j},\hat{k},\hat{l})+f_{12}(\hat{i},\hat{j},\hat{k},\hat{l})+f_{13}(\hat{i},\hat{j},\hat{k},\hat{l})] \\ &+[(y_1,y_2,i,k)(y_3,y_4,j,l)+(y_1,y_2,j,l)(y_3,y_4,i,k)] \\ &\times[f_{02}(\hat{i},\hat{j},\hat{k},\hat{l})+f_{12}(\hat{i},\hat{j},\hat{k},\hat{l})+f_{23}(\hat{i},\hat{j},\hat{k},\hat{l})] \\ &+[(y_1,y_2,i,l)(y_3,y_4,j,k)+(y_1,y_2,j,k)(y_3,y_4,i,l)] \\ &\times[f_{03}(\hat{i},\hat{j},\hat{k},\hat{l})+f_{13}(\hat{i},\hat{j},\hat{k},\hat{l})+f_{23}(\hat{i},\hat{j},\hat{k},\hat{l})]. \end{aligned} \tag{3.122}$$

Comparing the expression (3.117) for $h_0$ with the expression (3.122) for $h_1$, we obtain common terms $h_{01}$ of $h_0$ and $h_1$,

$$\begin{aligned} h_{01} = \sum_{5\le i<j<k<l\le 2n} &[(y_1,y_2,i,j)(y_3,y_4,k,l)+(y_1,y_2,k,l)(y_3,y_4,i,j)] \\ &\times f_{01}(\hat{i},\hat{j},\hat{k},\hat{l}) \\ &+[(y_1,y_2,i,k)(y_3,y_4,j,l)+(y_1,y_2,j,l)(y_3,y_4,i,k)] \\ &\times f_{02}(\hat{i},\hat{j},\hat{k},\hat{l}) \\ &+[(y_1,y_2,i,l)(y_3,y_4,j,k)+(y_1,y_2,j,k)(y_3,y_4,i,l)] \\ &\times f_{03}(\hat{i},\hat{j},\hat{k},\hat{l}). \end{aligned} \tag{3.123}$$

Let the decomposition of $h_1$ be

$$h_1 = h_{01} + \tilde{h}_1. \tag{3.124}$$

Then we find that

$$\begin{aligned} \tilde{h}_1 = \sum_{5\le i<j<k<l\le 2n} &[(y_1,y_2,i,j)(y_3,y_4,k,l)+(y_1,y_2,k,l)(y_3,y_4,i,j)] \\ &\times[f_{12}(\hat{i},\hat{j},\hat{k},\hat{l})+f_{13}(\hat{i},\hat{j},\hat{k},\hat{l})] \\ &+[(y_1,y_2,i,k)(y_3,y_4,j,l)+(y_1,y_2,j,l)(y_3,y_4,i,k)] \\ &\times[f_{12}(\hat{i},\hat{j},\hat{k},\hat{l})+f_{23}(\hat{i},\hat{j},\hat{k},\hat{l})] \\ &+[(y_1,y_2,i,l)(y_3,y_4,j,k)+(y_1,y_2,j,k)(y_3,y_4,i,l)] \\ &\times[f_{13}(\hat{i},\hat{j},\hat{k},\hat{l})+f_{23}(\hat{i},\hat{j},\hat{k},\hat{l})]. \end{aligned} \tag{3.125}$$

Interchanging $y_2$ with $y_3$ in Eqs.(3.117) and (3.122), we obtain common terms $h_{02}$ of $h_0$

and $h_2$,

$$\begin{aligned} h_{02} = \sum_{5\le i<j<k<l\le 2n} & [(y_1,y_3,i,j)(y_2,y_4,k,l)+(y_1,y_3,k,l)(y_2,y_4,i,j)] \\ & \times f_{01}(\hat{i},\hat{j},\hat{k},\hat{l}) \\ & +[(y_1,y_3,i,k)(y_2,y_4,j,l)+(y_1,y_3,j,l)(y_2,y_4,i,k)] \\ & \times f_{02}(\hat{i},\hat{j},\hat{k},\hat{l}) \\ & +[(y_1,y_3,i,l)(y_2,y_4,j,k)+(y_1,y_3,j,k)(y_2,y_4,i,l)] \\ & \times f_{03}(\hat{i},\hat{j},\hat{k},\hat{l}). \end{aligned} \tag{3.126}$$

Interchanging $y_2$ with $y_3$ in Eqs.(3.124) and (3.125), we obtain a decomposition of $h_2$,

$$h_2 = h_{02} + \tilde{h}_2, \tag{3.127}$$

where

$$\begin{aligned} \tilde{h}_2 = \sum_{5\le i<j<k<l\le 2n} & [(y_1,y_3,i,j)(y_2,y_4,k,l)+(y_1,y_3,k,l)(y_2,y_4,i,j)] \\ & \times [f_{12}(\hat{i},\hat{j},\hat{k},\hat{l})+f_{13}(\hat{i},\hat{j},\hat{k},\hat{l})] \\ & +[(y_1,y_3,i,k)(y_2,y_4,j,l)+(y_1,y_3,j,l)(y_2,y_4,i,k)] \\ & \times [f_{12}(\hat{i},\hat{j},\hat{k},\hat{l})+f_{23}(\hat{i},\hat{j},\hat{k},\hat{l})] \\ & +[(y_1,y_3,i,l)(y_2,y_4,j,k)+(y_1,y_3,j,k)(y_2,y_4,i,l)] \\ & \times [f_{13}(\hat{i},\hat{j},\hat{k},\hat{l})+f_{23}(\hat{i},\hat{j},\hat{k},\hat{l})]. \end{aligned} \tag{3.128}$$

Similarly we obtain common terms $h_{03}$ of $h_0$ and $h_3$ interchanging $y_2$ with $y_4$ in Eqs.(3.117) and (3.122)

$$\begin{aligned} h_{03} = \sum_{5\le i<j<k<l\le 2n} & [(y_1,y_4,i,j)(y_2,y_3,k,l)+(y_1,y_4,k,l)(y_2,y_3,i,j)] \\ & \times f_{01}(\hat{i},\hat{j},\hat{k},\hat{l}) \\ & +[(y_1,y_4,i,k)(y_2,y_3,j,l)+(y_1,y_4,j,l)(y_2,y_3,i,k)] \\ & \times f_{02}(\hat{i},\hat{j},\hat{k},\hat{l}) \\ & +[(y_1,y_4,i,l)(y_2,y_3,j,k)+(y_1,y_4,j,k)(y_2,y_3,i,l)] \\ & \times f_{03}(\hat{i},\hat{j},\hat{k},\hat{l}). \end{aligned} \tag{3.129}$$

Interchanging $y_2$ with $y_4$ in Eqs.(3.124) and (3.125), we obtain a decomposition of $h_3$,

$$h_3 = h_{03} + \tilde{h}_3, \tag{3.130}$$

where

$$\begin{aligned}\tilde{h}_3 = \sum_{5\le i<j<k<l\le 2n} &[(y_1,y_4,i,j)(y_2,y_3,k,l)+(y_1,y_4,k,l)(y_2,y_3,i,j)]\\ &\quad\times[f_{12}(\hat{i},\hat{j},\hat{k},\hat{l})+f_{13}(\hat{i},\hat{j},\hat{k},\hat{l})]\\ &+[(y_1,y_4,i,k)(y_2,y_3,j,l)+(y_1,y_4,j,l)(y_2,y_3,i,k)]\\ &\quad\times[f_{12}(\hat{i},\hat{j},\hat{k},\hat{l})+f_{23}(\hat{i},\hat{j},\hat{k},\hat{l})]\\ &+[(y_1,y_4,i,l)(y_2,y_3,j,k)+(y_1,y_4,j,k)(y_2,y_3,i,l)]\\ &\quad\times[f_{13}(\hat{i},\hat{j},\hat{k},\hat{l})+f_{23}(\hat{i},\hat{j},\hat{k},\hat{l})].\end{aligned}\tag{3.131}$$

We find, using the expansion formulae (3.114), (3.115), (3.116) and the assumption (3.113), that a simple sum of $h_{01}, h_{02}$ and $h_{03}$ is reduced to $h_0$,

$$\begin{aligned}&h_{01}+h_{02}+h_{03}\\ &=\sum_{5\le i<j<k<l\le 2n}(y_1,y_2,y_3,y_4,i,j,k.l)\\ &\qquad\times[f_{01}(\hat{i},\hat{j},\hat{k},\hat{l})+f_{02}(\hat{i},\hat{j},\hat{k},\hat{l})+f_{03}(\hat{i},\hat{j},\hat{k},\hat{l})],\\ &=\sum_{5\le i<j<k<l\le 2n}(y_1,y_2,y_3,y_4,i,j,k.l)(\hat{i},\hat{j},\hat{k},\hat{l})_{n-2}(\cdots)_{n-2}\\ &=h_0.\end{aligned}\tag{3.132}$$

Accordingly $h_0, h_1, h_2$ and $h_3$ are decomposed into

$$h_0 = h_{01}+h_{02}+h_{03},\tag{3.133}$$

$$h_1 = h_{01}+\tilde{h}_1,\tag{3.134}$$

$$h_2 = h_{02}+\tilde{h}_2,\tag{3.135}$$

$$h_3 = h_{03}+\tilde{h}_3,\tag{3.136}$$

## § 3.5. Expressions for $\tilde{h}_1, \tilde{h}_2$ and $\tilde{h}_3$

We rearrange $\tilde{h}_1, \tilde{h}_2$ and $\tilde{h}_3$ as

$$\begin{aligned}\tilde{h}_1 = \sum_{5\le i<j<k<l\le 2n} &[(y_1,y_2,i,j)(y_3,y_4,k,l)+(y_1,y_2,k,l)(y_3,y_4,i,j)\\ &\quad+(y_1,y_2,i,k)(y_3,y_4,j,l)+(y_1,y_2,j,l)(y_3,y_4,i,k)]\ f_{12}(\hat{i},\hat{j},\hat{k},\hat{l})\\ &+[(y_1,y_2,i,j)(y_3,y_4,k,l)+(y_1,y_2,k,l)(y_3,y_4,i,j)\\ &\quad+(y_1,y_2,i,l)(y_3,y_4,j,k)+(y_1,y_2,j,k)(y_3,y_4,i,l)]\ f_{13}(\hat{i},\hat{j},\hat{k},\hat{l})\\ &+[(y_1,y_2,i,k)(y_3,y_4,j,l)+(y_1,y_2,j,l)(y_3,y_4,i,k)\\ &\quad+(y_1,y_2,i,l)(y_3,y_4,j,k)+(y_1,y_2,j,k)(y_3,y_4,i,l)]\ f_{23}(\hat{i},\hat{j},\hat{k},\hat{l}),\end{aligned}\tag{3.137}$$

$$\tilde{h}_2 = \sum_{5\leq i<j<k<l\leq 2n} [(y_1,y_3,i,j)(y_2,y_4,k,l)+(y_1,y_3,k,l)(y_2,y_4,i,j)$$
$$+(y_1,y_3,i,k)(y_2,y_4,j,l)+(y_1,y_3,j,l)(y_2,y_4,i,k)]\ f_{12}(\hat{i},\hat{j},\hat{k},\hat{l})$$
$$+[(y_1,y_3,i,j)(y_2,y_4,k,l)+(y_1,y_3,k,l)(y_2,y_4,i,j)$$
$$+(y_1,y_3,i,l)(y_2,y_4,j,k)+(y_1,y_3,j,k)(y_2,y_4,i,l)]\ f_{13}(\hat{i},\hat{j},\hat{k},\hat{l})$$
$$+[(y_1,y_3,i,k)(y_2,y_4,j,l)+(y_1,y_3,j,l)(y_2,y_4,i,k)$$
$$+(y_1,y_3,i,l)(y_2,y_4,j,k)+(y_1,y_3,j,k)(y_2,y_4,i,l)]\ f_{23}(\hat{i},\hat{j},\hat{k},\hat{l})$$

(3.138)

and

$$\tilde{h}_3 = \sum_{5\leq i<j<k<l\leq 2n} [(y_1,y_4,i,j)(y_2,y_3,k,l)+(y_1,y_4,k,l)(y_2,y_3,i,j)$$
$$+(y_1,y_4,i,k)(y_2,y_3,j,l)+(y_1,y_4,j,l)(y_2,y_3,i,k)]\ f_{12}(\hat{i},\hat{j},\hat{k},\hat{l})$$
$$+[(y_1,y_4,i,j)(y_2,y_3,k,l)+(y_1,y_4,k,l)(y_2,y_3,i,j)$$
$$+(y_1,y_4,i,l)(y_2,y_3,j,k)+(y_1,y_4,j,k)(y_2,y_3,i,l)]\ f_{13}(\hat{i},\hat{j},\hat{k},\hat{l})$$
$$+[(y_1,y_4,i,k)(y_2,y_3,j,l)+(y_1,y_4,j,l)(y_2,y_3,i,k)$$
$$+(y_1,y_4,i,l)(y_2,y_3,j,k)+(y_1,y_4,j,k)(y_2,y_3,i,l)]\ f_{23}(\hat{i},\hat{j},\hat{k},\hat{l}).$$

(3.139)

We have identities of hafnians, Eqs.(2.23),(2.24) and (2.25), which are written as

$$\begin{aligned}&(y_1,y_2,i,j)(y_3,y_4,k,l)+(y_1,y_2,k,l)(y_3,y_4,i,j)\\&+(y_1,y_2,i,k)(y_3,y_4,j,l)+(y_1,y_2,j,l)(y_3,y_4,i,k)\\&=(y_1,y_3,i,l)(y_2,y_4,j,k)+(y_1,y_3,j,k)(y_2,y_4,i,l)\\&+(y_1,y_4,i,l)(y_2,y_3,j,k)+(y_1,y_4,j,k)(y_2,y_3,i,l),\end{aligned} \tag{3.140}$$

$$\begin{aligned}&(y_1,y_2,i,j)(y_3,y_4,k,l)+(y_1,y_2,k,l)(y_3,y_4,i,j)\\&+(y_1,y_2,i,l)(y_3,y_4,j,k)+(y_1,y_2,j,k)(y_3,y_4,i,l)\\&=(y_1,y_3,i,k)(y_2,y_4,j,l)+(y_1,y_3,j,l)(y_2,y_4,i,k)\\&+(y_1,y_4,i,k)(y_2,y_3,j,l)+(y_1,y_4,j,l)(y_2,y_3,i,k)\end{aligned} \tag{3.141}$$

and

$$\begin{aligned}&(y_1,y_2,i,k)(y_3,y_4,j,l)+(y_1,y_2,j,l)(y_3,y_4,i,k)\\&+(y_1,y_2,i,l)(y_3,y_4,j,k)+(y_1,y_2,j,k)(y_3,y_4,i,l)\\&=(y_1,y_3,i,j)(y_2,y_4,k,l)+(y_1,y_3,k,l)(y_2,y_4,i,j)\\&+(y_1,y_4,i,j)(y_2,y_3,k,l)+(y_1,y_4,k,l)(y_2,y_3,i,j).\end{aligned} \tag{3.142}$$

Substituting these identities (3.140),(3.141) and (3.142) into Eq.(3.137) we obtain an expression for $\tilde{h}_1$,

$$\begin{aligned}\tilde{h}_1 = \sum_{5\le i<j<k<l\le 2n} &[(y_1,y_3,i,l)(y_2,y_4,j,k)+(y_1,y_3,j,k)(y_2,y_4,i,l)\\ &\quad +[(y_1,y_4,i,l)(y_2,y_3,j,k)+(y_1,y_4,j,k)(y_2,y_3,i,l)]\ f_{12}(\hat{i},\hat{j},\hat{k},\hat{l})\\ &+[(y_1,y_3,i,k)(y_2,y_4,j,l)+(y_1,y_3,j,l)(y_2,y_4,i,k)\\ &\ +(y_1,y_4,i,k)(y_2,y_3,j,l)+(y_1,y_4,j,l)(y_2,y_3,i,k)]\ f_{13}(\hat{i},\hat{j},\hat{k},\hat{l})\\ &+[(y_1,y_3,i,j)(y_2,y_4,k,l)+(y_1,y_3,k,l)(y_2,y_4,i,j)\\ &\ +(y_1,y_4,i,j)(y_2,y_3,k,l)+(y_1,y_4,k,l)(y_2,y_3,i,j)]\ f_{23}(\hat{i},\hat{j},\hat{k},\hat{l}).\end{aligned} \tag{3.143}$$

Interchanging $y_2$ with $y_3$ in Eq.(3.143) we obtain an expression for $\tilde{h}_2$,

$$\begin{aligned}\tilde{h}_2 = \sum_{5\le i<j<k<l\le 2n} &[(y_1,y_2,i,l)(y_3,y_4,j,k)+(y_1,y_2,j,k)(y_3,y_4,i,l)\\ &\quad +[(y_1,y_4,i,l)(y_2,y_3,j,k)+(y_1,y_4,j,k)(y_2,y_3,i,l)]\ f_{12}(\hat{i},\hat{j},\hat{k},\hat{l})\\ &+[(y_1,y_2,i,k)(y_3,y_4,j,l)+(y_1,y_2,j,l)(y_3,y_4,i,k)\\ &\ +(y_1,y_4,i,k)(y_2,y_3,j,l)+(y_1,y_4,j,l)(y_2,y_3,i,k)]\ f_{13}(\hat{i},\hat{j},\hat{k},\hat{l})\\ &+[(y_1,y_2,i,j)(y_3,y_4,k,l)+(y_1,y_2,k,l)(y_3,y_4,i,j)\\ &\ +(y_1,y_4,i,j)(y_2,y_3,k,l)+(y_1,y_4,k,l)(y_2,y_3,i,j)]\ f_{23}(\hat{i},\hat{j},\hat{k},\hat{l}).\end{aligned} \tag{3.144}$$

Similarly interchanging $y_2$ with $y_4$ in Eq.(3.143) we obtain an expression for $\tilde{h}_3$,

$$\begin{aligned}\tilde{h}_3 = \sum_{5\le i<j<k<l\le 2n} &[(y_1,y_3,i,l)(y_2,y_4,j,k)+(y_1,y_3,j,k)(y_2,y_4,i,l)\\ &\quad +[(y_1,y_2,i,l)(y_3,y_4,j,k)+(y_1,y_2,j,k)(y_3,y_4,i,l)]\ f_{12}(\hat{i},\hat{j},\hat{k},\hat{l})\\ &+[(y_1,y_3,i,k)(y_2,y_4,j,l)+(y_1,y_3,j,l)(y_2,y_4,i,k)\\ &\ +(y_1,y_2,i,k)(y_3,y_4,j,l)+(y_1,y_2,j,l)(y_3,y_4,i,k)]\ f_{13}(\hat{i},\hat{j},\hat{k},\hat{l})\\ &+[(y_1,y_3,i,j)(y_2,y_4,k,l)+(y_1,y_3,k,l)(y_2,y_4,i,j)\\ &\ +(y_1,y_2,i,j)(y_3,y_4,k,l)+(y_1,y_2,k,l)(y_3,y_4,i,j)]\ f_{23}(\hat{i},\hat{j},\hat{k},\hat{l}).\end{aligned} \tag{3.145}$$

From the expressions (3.143),(3.144) and (3.145) for $\tilde{h}_1$, $\tilde{h}_2$ and $\tilde{h}_3$, respectively we obtain the following decompositions,

$$\tilde{h}_1 = h_{12} + h_{13}, \tag{3.146}$$

$$\tilde{h}_2 = h_{12} + h_{23}, \tag{3.147}$$

$$\tilde{h}_3 = h_{13} + h_{23}, \tag{3.148}$$

where

$$\begin{aligned} h_{12} = \sum_{5\le i<j<k<l\le 2n} & [(y_1,y_4,i,l)(y_2,y_3,j,k) + (y_1,y_4,j,k)(y_2,y_3,i,l)]\ f_{12}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l}) \\ & +[(y_1,y_4,i,k)(y_2,y_3,j,l) + (y_1,y_4,j,l)(y_2,y_3,i,k)]\ f_{13}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l}) \\ & +[(y_1,y_4,i,j)(y_2,y_3,k,l) + (y_1,y_4,k,l)(y_2,y_3,i,j)]\ f_{23}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l}), \end{aligned} \tag{3.149}$$

$$\begin{aligned} h_{13} = \sum_{5\le i<j<k<l\le 2n} & [(y_1,y_3,i,l)(y_2,y_4,j,k) + (y_1,y_3,j,k)(y_2,y_4,i,l)]\ f_{12}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l}) \\ & +[(y_1,y_3,i,k)(y_2,y_4,j,l) + (y_1,y_3,j,l)(y_2,y_4,i,k)]\ f_{13}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l}) \\ & +[(y_1,y_3,i,j)(y_2,y_4,k,l) + (y_1,y_3,k,l)(y_2,y_4,i,j)]\ f_{23}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l}) \end{aligned} \tag{3.150}$$

and

$$\begin{aligned} h_{23} = \sum_{5\le i<j<k<l\le 2n} & [(y_1,y_2,i,l)(y_3,y_4,j,k) + (y_1,y_2,j,k)(y_3,y_4,i,l)]\ f_{12}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l}) \\ & +[(y_1,y_2,i,k)(y_3,y_4,j,l) + (y_1,y_2,j,l)(y_3,y_4,i,k)]\ f_{13}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l}) \\ & +[(y_1,y_2,i,j)(y_3,y_4,k,l) + (y_1,y_2,k,l)(y_3,y_4,i,j)]\ f_{23}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l}). \end{aligned} \tag{3.151}$$

## § 3.6. Decompositions of $f_0, f_1, f_2$ and $f_3$

We have found in the previous subsections the decompositions,

$$h_1 = h_{01} + \tilde{h}_1, \quad h_2 = h_{02} + \tilde{h}_2, \quad h_3 = h_{03} + \tilde{h}_3, \tag{3.152}$$

$$\tilde{h}_1 = h_{12} + h_{13}, \quad \tilde{h}_2 = h_{12} + h_{23}, \quad \tilde{h}_3 = h_{13} + h_{33}, \tag{3.153}$$

which together with Eqs.(3.133), give the decompositions of $h_0, h_1, h_2$ and $h_3$,

$$h_0 = h_{01} + h_{02} + h_{03}, \quad h_1 = h_{01} + h_{12} + h_{13}, \tag{3.154}$$

$$h_2 = h_{02} + h_{12} + h_{23}, \quad h_3 = h_{03} + h_{13} + h_{23}. \tag{3.155}$$

On the other hand, we have found the decompositions,

$$f_0 = \overline{f}_1 + \overline{f}_2 + \overline{f}_3 + h_0, \tag{3.156}$$

$$f_1 = \overline{f}_1 + g_1, \quad g_1 = g_{12} + g_{13} + h_1, \tag{3.157}$$

$$f_2 = \overline{f}_2 + g_2, \quad g_2 = g_{12} + g_{23} + h_2, \tag{3.158}$$

$$f_3 = \overline{f}_3 + g_3, \quad g_3 = g_{13} + g_{23} + h_3. \tag{3.159}$$

Combining these decompositions we finally obtain the decompositions,

$$f_0 = f_{01} + f_{02} + f_{03}, \quad f_1 = f_{01} + f_{12} + f_{13}, \tag{3.160}$$

$$f_2 = f_{02} + f_{12} + f_{23}, \quad f_3 = f_{03} + f_{13} + f_{23}, \tag{3.161}$$

where

$$f_{01} = \overline{f}_1 + h_{01}, \quad f_{02} = \overline{f}_2 + h_{02}, \quad f_{03} = \overline{f}_3 + h_{03}, \tag{3.162}$$

$$f_{12} = g_{12} + h_{12}, \quad f_{13} = g_{13} + h_{13}, \quad f_{23} = g_{23} + h_{23}. \tag{3.163}$$

Thus we have proved that the decompositions of the products of $n$th-order hafnians,

$$f_0 = (1, 2, 3, 4, 5, 6, \cdots, 2n)(5, 6, \cdots, 2n) \tag{3.164}$$

$$f_1 = (1, 2, 5, 6, \cdots, 2n)(3, 4, 5, 6, \cdots, 2n) \tag{3.165}$$

$$f_2 = (1, 3, 5, 6, \cdots, 2n)(2, 4, 5, 6, \cdots, 2n) \tag{3.166}$$

$$f_3 = (1, 3, 5, 6, \cdots, 2n)(2, 4, 5, 6, \cdots, 2n), \tag{3.167}$$

provided that the decompositions of the products of $(n-2)$th-order hafnians hold. It is easily proved that the decompositions of the products of fourth-order hafnians hold. Hence we have proved by induction that the decompositons of the products of any order hafnians.

## § 4. Ultradiscrete analogue of the identity of pfaffians

It is known that a variety of soliton equations exhibiting multi-soliton solutions expressed by pfaffians give rise to the following identity of pfaffians,

$$\begin{aligned}
&\mathrm{pf}(1, 2, 3, 4, 5, 6, \cdots, 2n)\mathrm{pf}(5, 6, \cdots, 2n) \\
&= \mathrm{pf}(1, 2, 5, 6, \cdots, 2n)\mathrm{pf}(3, 4, 5, 6, \cdots, 2n) \\
&-\mathrm{pf}(1, 3, 5, 6, \cdots, 2n)\mathrm{pf}(2, 4, 5, 6, \cdots, 2n) \\
&+\mathrm{pf}(1, 4, 5, 6, \cdots, 2n)\mathrm{pf}(2, 3, 5, 6, \cdots, 2n).
\end{aligned} \tag{4.1}$$

It is known that the pfaffians can not be ultradiscretized due to negativity terms. A remedy for the problem was found by Takahashi and the author of the present article.

They have expressed the multi-soliton solutions to an ultradiscretized soliton equation called "Box and Ball System" by ultradiscretized *permanents* instead of determinants. Accordingly we consider an ultradiscrete form of the identity of hafnians.

We have

$$f_0 = (1,2,3,4,5,6,\cdots,2n)(5,6,\cdots,2n), \tag{4.2}$$
$$f_1 = (1,2,5,6,\cdots,2n)(3,4,5,6,\cdots,2n), \tag{4.3}$$
$$f_2 = (1,3,5,6,\cdots,2n)(2,4,5,6,\cdots,2n), \tag{4.4}$$
$$f_3 = (1,4,5,6,\cdots,2n)(2,3,5,6,\cdots,2n), \tag{4.5}$$

which were decomposed in the previous section into the following form,

$$f_0 = f_{01} + f_{02} + f_{03}, \tag{4.6}$$
$$f_1 = f_{01} + f_{12} + f_{13}, \tag{4.7}$$
$$f_2 = f_{02} + f_{12} + f_{23}, \tag{4.8}$$
$$f_3 = f_{03} + f_{13} + f_{23}. \tag{4.9}$$

We consider a relation of hafnians,

$$f_0 + f_2 = f_1 + f_3, \tag{4.10}$$

which does holds for pfaffians but not for hafnians.

Let

$$f_0 = \exp F_0/\epsilon, \quad f_1 = \exp F_1/\epsilon, \quad f_2 = \exp F_2/\epsilon, \quad f_3 = \exp F_3/\epsilon, \tag{4.11}$$
$$f_{01} = \exp F_{01}/\epsilon, \quad f_{02} = \exp F_{02}/\epsilon, \quad f_{03} = \exp F_{03}/\epsilon, \tag{4.12}$$
$$f_{12} = \exp F_{12}/\epsilon \quad f_{13} = \exp F_{13}/\epsilon, \quad f_{23} = \exp F_{23}/\epsilon. \tag{4.13}$$

Taking a limit $\epsilon \to +0$ in Eq.(4.10), we have an ultradiscrete form of the identity of the hafnians,

$$\max(F_0, F_2) = \max(F_1, F_3). \tag{4.14}$$

By virtue of the decomposition of the hafnians we find that

$$F_0 = \max(F_{01}, F_{02}, F_{03}), \tag{4.15}$$
$$F_1 = \max(F_{01}, F_{12}, F_{13}), \tag{4.16}$$
$$F_2 = \max(F_{02}, F_{12}, F_{23}), \tag{4.17}$$
$$F_3 = \max(F_{03}, F_{13}, F_{23}), \tag{4.18}$$

and Eq.(4.14) is expressed by

$$\max(F_{01}, F_{02}, F_{03}, F_{12}, F_{23}) = \max(F_{01}, F_{03}, F_{12}, F_{13}, F_{23}). \tag{4.19}$$

We shall investigate whether Eq.(4.19) does hold or not. Equation (4.19) holds in the following six cases,

(i) $F_{01} \geq \max(F_{02}, F_{03}, F_{12}, F_{13}, F_{23})$,
(ii) $F_{02} = \max(F_{01}, F_{03}, F_{12}, F_{13}, F_{23})$,
(iii) $F_{03} \geq \max(F_{01}, F_{02}, F_{12}, F_{13}, F_{23})$,
(iv) $F_{12} \geq \max(F_{01}, F_{02}, F_{03}, F_{13}, F_{23})$,
(v) $F_{13} = \max(F_{01}, F_{02}, F_{03}, F_{12}, F_{23})$,
(vi) $F_{23} \geq \max(F_{01}, F_{02}, F_{03}, F_{12}, F_{13})$,

except for the following two cases,

(vii) $F_{02} > \max(F_{01}, F_{03}, F_{12}, F_{13}, F_{23})$,
(viii) $F_{13} > \max(F_{01}, F_{02}, F_{03}, F_{12}, F_{23})$.

In the cases (vii) and (viii) we have the relation, $F_0 = F_2$ and $F_1 = F_3$ respectively. Accordingly we find the algebraic identiy of the ultradiscretized hafnians,

$$(\max(F_0, F_2) - \max(F_1, F_3))(F_0 - F_2)(F_1 - F_3) = 0, \tag{4.20}$$

where $F_0, F_1, F_2$ and $F_3$ are the ultradiscrete form of $f_0, f_1, f_2$ and $f_3$,respectively.

We have thus proved the ultradiscrete analogue of the identity of the pfaffians.

## References


[1] Ryogo Hirota, *The direct method in soliton theory*, Cambridge tracts in mathematics;155, Cambridge University Press, 2004.
[2] D.Takahashi and R.Hirota, "Ultradiscrete Soliton Solution of Permanent Type", J. Phy. Soc. Jpn **76** (2007) 104007.
[3] H.Nagai, "A New Expression of Soliton Solution to the Ultradiscrete Toda Equation", J. Phys. A: Math. Theor. **41** (2008) 235204.
[4] E.R.Caieniello, *Combinatorics and renormalization in quantum field theory*, Frontiers in Physics. W.A.Benjamin, Inc., Reading, Mass.-London-Amsterdam, 1973.
[5] T.Tokihiro, D.Takahashi, J.Matsukidaira, and J.Satsuma, "From Soliton Equations to Integrable Cellular Automata through a Limiting Procedure" Phys. Rev. Lett. **76** (1996) 3247.